LBL-30673

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

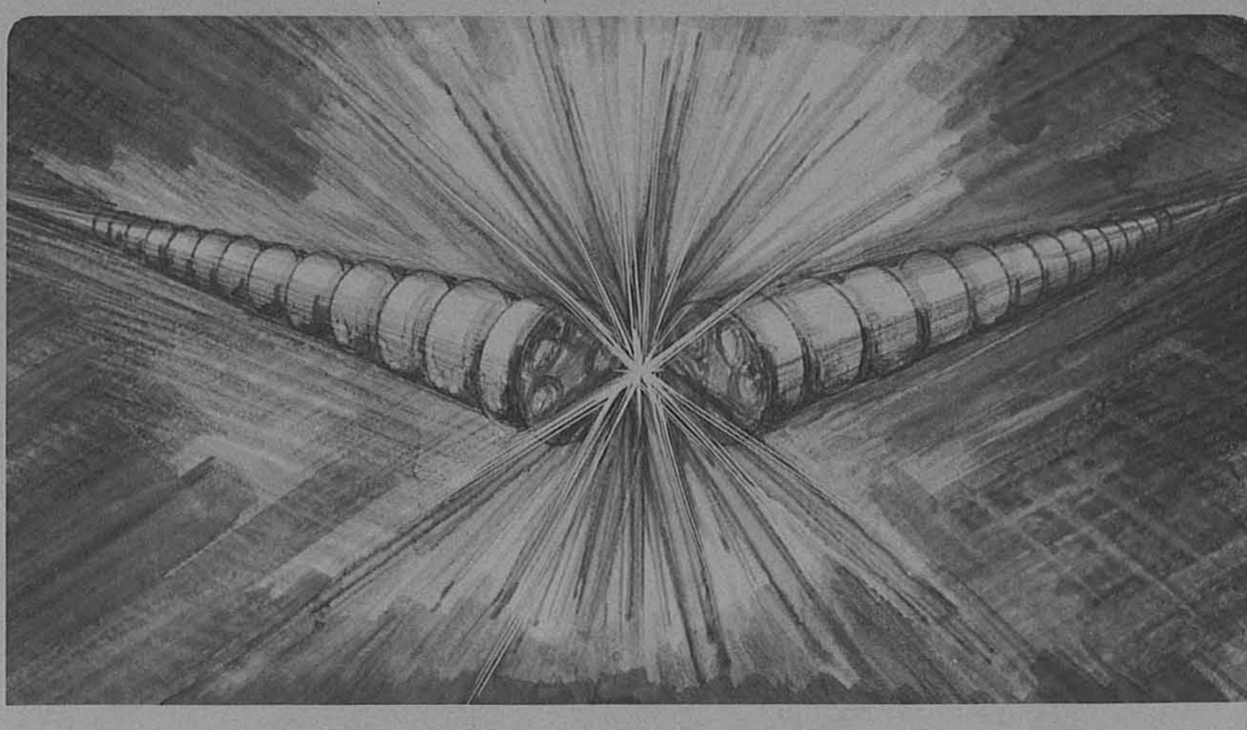
Accelerator & Fusion Research Division

Presented at the Thirteenth International Free Electron Laser Conference, Santa Fe, NM, August 25–30, 1991, and to be published in the Proceedings

Three-Dimensional Free Electron Laser Dispersion Relation Including Betatron Oscillations

Y.H. Chin, K.-J. Kim, and M. Xie

August 1991



Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098

DISCLAIMER

LBL-30673
ESG-132

THREE-DIMENSIONAL FREE ELECTRON LASER DISPERSION RELATION INCLUDING BETATRON OSCILLATIONS*

Yong Ho Chin, Kwang-Je Kim, and Ming Xie

Lawrence Berkeley Laboratory
University of California
Berkeley, CA 94720

Presented at the Thirteenth International Free Electron Laser Conference, held in Santa Fe, New-Mexico, August 25-30, 1991, and to be published in the Proceedings.

* This work was supported by the Director, Office of Energy Research, Office of Basic Energy Sciences, Materials Sciences Division, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

Three-Dimensional Free Electron Laser Dispersion Relation Including Betatron Oscillations *

Yong Ho Chin, Kwang-Je Kim, and Ming Xie

Lawrence Berkeley Laboratory,
University of California
Berkeley, CA 94720

Abstract

We have developed a 3-D FEL theory based upon the Maxwell-Vlasov equations including the effects of the energy spread and emittance of the electron beam, and of betatron oscillations. The radiation field is expressed in terms of the Green's function of the inhomogeneous wave equation and the distribution function of the electron beam. The distribution function is expanded in terms of a set of orthogonal functions determined by the unperturbed particle distribution. The coupled Maxwell-Vlasov equations are then reduced to a matrix equation, from which a dispersion relation for the eigenvalues is derived. In the limit of small betatron oscillation frequency, the present dispersion relation reduces to the well-known cubic equation of the one-dimensional theory in the limit of large beam size, and it gives the correct gain in the limit of small beam size. Comparisons of our numerical results with other approaches show good agreement. We present a handy empirical formula for the FEL gain of a 3-D Gaussian beam, as a function of the scaled parameters, that can be used for a quick estimate of the gain.

I. Introduction

It is widely known that transverse emittance and betatron oscillation can significantly reduce the gain in a Free Electron Laser(FEL) operating in the high gain regime before saturation, due to a spread in the longitudinal velocity of electrons. One approach to study these effects is based on an integro-differential eigenvalue equation involving the radiation field alone, derived by reducing the coupled Maxwell-Vlasov equations[1]. However, inclusion of the emittance and betatron oscillation effects makes it extremely

* This work was supported by the Director, Office of Energy Research, Office of Basic Energy Sciences, Materials Sciences Division, the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

difficult to solve the equation exactly. Krinsky, Yu and Gluckstern[2] have proposed the variational method to solve the equation approximately. Giving up on seeking an exact eigenfunction, they replace it by a trial function and concentrate on solutions of eigenvalues. The principle behind this method is the fact that the error in the eigenvalue depends quadratically on errors in the trial function. However, one must prepare a "good" trial function, and in general, the accuracy of calculation is unknown.

In this paper, we present a new approach based on an orthogonal expansion of the electron distribution function. Starting with the Maxwell-Vlasov equations and equations of motion for an electron, we combine them into a single integral equation for the *electron distribution function.* The radiation field is expressed explicitly in terms of the Green's function of the inhomogeneous wave equation and the electron distribution function. The distribution function is then expanded around the unperturbed part in terms of a set of orthogonal functions determined by the shape of the unperturbed part. This expansion converts the integral equation into a matrix equation, from which a dispersion relation for the eigenvalues is derived. This dispersion relation has a form similar to that in plasma physics. The present method has the advantage that the higher-order terms in the expansion can be determined in a systematic fashion. It turns out that the series expansion of the perturbed distribution function by an appropriate set of orthogonal functions converges very quickly. As a matter of fact, one can obtain an accurate eigenvalue by taking only the lowest-order expansion term. Due to space limitations, here we briefly describe only the outline of the formulation and the results. Details of the formulation will be presented in a subsequent paper.

II. Vlasov Equation

We consider the electron beam moving in the z-direction with average energy γ_r through a periodic helical wiggler with wave number k_w and field strength parameter K. We choose z, the distance from the wiggler entrance, as the independent variable. After averaging over the fast wiggling motion, the transverse electron motion can be described by the harmonic betatron oscillation in the transverse vector $\boldsymbol{x}_\beta$. The transverse coordinates to be used in the Vlasov equation are $\boldsymbol{x}_\beta$ and its canonical momentum conjugate, $\boldsymbol{p}_\beta$. The longitudinal variables are τ, the relative position of an electron from the resonant electron (in time units), and the electron energy γ.

The linearized Vlasov equation for the perturbed part of the distribution function,

$f_1(\boldsymbol{x}_\beta, \boldsymbol{p}_\beta, \tau, \gamma; z)$, is then written as

$$\frac{\partial f_1}{\partial z} + \boldsymbol{p}_\beta \frac{\partial f_1}{\partial \boldsymbol{x}_\beta} - k_\beta^2 \boldsymbol{x}_\beta \frac{\partial f_1}{\partial \boldsymbol{p}_\beta} + \frac{d\tau}{dz}\frac{\partial f_1}{\partial \tau} + \frac{d\gamma}{dz}\frac{\partial f_0}{\partial \gamma} = 0, \tag{1}$$

where f_0 is the unperturbed electron distribution. In this paper, we assume that the focusing in the wiggler is matched so that f_0 is a function of $\boldsymbol{p}_\beta{}^2 + k_\beta^2 \boldsymbol{x}_\beta{}^2$ and γ only, and we also assume for simplicity that f_0 can be factorized as:

$$f_0 = f_{0\perp}(\boldsymbol{p}_\beta{}^2 + k_\beta^2 \boldsymbol{x}_\beta{}^2) \cdot f_{0\|}(\gamma), \tag{2}$$

where f_0 is normalized so that its integral over six dimensional phase space is equal to the total number of electrons. The equation of motion of τ is given by

$$\frac{d\tau}{dz} = \frac{1}{c}[-2\frac{k_w}{k_1}\frac{\gamma - \gamma_r}{\gamma_r} + \frac{1}{2}(\boldsymbol{p}_\beta{}^2 + k_\beta^2 \boldsymbol{x}_\beta{}^2)], \tag{3}$$

where c is the speed of light, $k_1 = 2k_w\gamma_r^2/(1+K^2)$, and k_β is the betatron wave number. (In the absence of external focusing, $k_\beta = Kk_w/\gamma\sqrt{2}$). The energy change is produced by the interaction of the electron's helical motion and the radiation field. The vector potential $\boldsymbol{A}_R(\boldsymbol{x}, z, t)$ for the radiation field satisfies the inhomogeneous wave equation, and its solution can be written using the Green's function of the wave equation and the electron charge density corresponding to f_1. After a lengthy calculation, we obtain the expression for the energy change

$$\frac{d\gamma}{dz} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\frac{1}{2\pi i}\int_{q_0-i\infty}^{q_0+i\infty}[\int_{-\infty}^{\infty} P_{\omega q}(k_\perp)\rho_{\omega q}(\boldsymbol{k}_\perp)e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\beta}d^2\boldsymbol{k}_\perp]e^{qz}dq\}e^{-i\omega\tau}d\omega, \tag{4}$$

where

$$P_{\omega q}(k_\perp) = -\frac{r_e}{2\pi c}(\frac{K}{\gamma_r})^2\frac{(\mathrm{J}_1(k_\perp r_h)/(k_\perp r_h))^2 + \mathrm{J}_1'^2(k_\perp r_h)}{\sqrt{1-(k_\perp/k)^2}[q + i(\frac{\omega}{v_r} - \sqrt{k^2 - k_\perp^2} - k_w)]}, \tag{5}$$

where $k = \omega/c$, $k_\perp = |\boldsymbol{k}_\perp|$, r_e is the classical electron radius, r_h is the radius of the helical motion, $\mathrm{J}_1(x)$ and $\mathrm{J}_1'(x)$ are the Bessel function and its derivative, respectively, and $\rho_{\omega q}(\boldsymbol{k}_\perp)$ is the Laplace-Fourier transform of the charge density, which is related to $f_1(\boldsymbol{x}_\beta, \boldsymbol{p}_\beta, \tau, \gamma; z)$ by

$$\rho_{\omega q}(\boldsymbol{k}_\perp) = \int_{-\infty}^{\infty}\{\int_0^{\infty}[\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}\int_0^{\infty} f_1(\boldsymbol{x}_\beta, \boldsymbol{p}_\beta, \tau, \gamma; z)d^2\boldsymbol{p}_\beta d\gamma)e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\beta}d^2\boldsymbol{x}_\beta]e^{-qz}dz\}e^{i\omega\tau}d\tau. \tag{6}$$

Equation(5) is an exact equation. The denominator may be well approximated by $[q + ik_w(k-k_1)/k_1 + ik\theta^2/2]$ where $\theta = \tan^{-1}(k_\perp/k)$.

If we substitute Eqs.(3) and (4) into Eq.(1) and take its Fourier-Laplace transform, the linearized Vlasov equation becomes

$$[q - i\omega\frac{d\tau}{dz}]f_{\omega q} + \boldsymbol{p}_\beta\frac{\partial f_{\omega q}}{\partial \boldsymbol{x}_\beta} - k_\beta^2\boldsymbol{x}_\beta\frac{\partial f_{\omega q}}{\partial \boldsymbol{p}_\beta} = -f_{0\perp}\frac{df_{0\parallel}}{d\gamma}\int P_{\omega q}(k_\perp)\rho_{\omega q}(\boldsymbol{k}_\perp)e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\beta}d^2\boldsymbol{k}_\perp, \quad (7)$$

where $f_{\omega q}$ is the Fourier-Laplace transform of $f_1(\boldsymbol{x}_\beta, \boldsymbol{p}_\beta, \tau, \gamma; z)$

$$f_{\omega q}(\boldsymbol{x}_\beta, \boldsymbol{p}_\beta, \gamma) = \int_{-\infty}^{\infty}\{\int_0^\infty f_1(\boldsymbol{x}_\beta, \boldsymbol{p}_\beta, \tau, \gamma; z)e^{-qz}dz\}e^{i\omega\tau}d\tau. \quad (8)$$

We have ignored the Fourier transform of the initial distribution at $z = 0$ in Eq.(7) for simplicity. (If we retain this term, the problem becomes an initial value problem.)

Since the betatron motion of the electron is a simple harmonic oscillation, it is natural to introduce polar-coordinates in the transverse planes as

$$x_\beta = r_x\cos\phi_x, \quad y_\beta = r_y\cos\phi_y, \quad (9)$$

$$\frac{p_{x\beta}}{k_\beta} = r_x\sin\phi_x, \quad \frac{p_{y\beta}}{k_\beta} = r_y\sin\phi_y. \quad (10)$$

Then, the second and third terms in the LHS of the Vlasov equation, Eq.(7), are written as

$$\boldsymbol{p}_\beta\frac{\partial f_{\omega q}}{\partial \boldsymbol{x}_\beta} - k_\beta^2\boldsymbol{x}_\beta\frac{\partial f_{\omega q}}{\partial \boldsymbol{p}_\beta} = -k_\beta(\frac{\partial f_{\omega q}}{\partial\phi_x} + \frac{\partial f_{\omega q}}{\partial\phi_y}). \quad (11)$$

Now, due to the periodic boundary condition for $f_{\omega q}$ in the azimuthal angles ϕ_x and ϕ_y, $f_{\omega q}$ can be Fourier decomposed with respect to ϕ_x and ϕ_y into an infinite series of modes:

$$f_{\omega q}(\boldsymbol{x}_\beta, \boldsymbol{p}_\beta, \gamma) = \sum_{m,n=-\infty}^{\infty} F_{\omega q}^{(m,n)}(r_x, r_y, \gamma)e^{im\phi_x}e^{in\phi_y}, \quad (12)$$

where m and n are integers. Combining Eqs.(6), (7), and (12), we obtain an integral equation for $F_{\omega q}^{(m,n)}$

$$\begin{aligned}[q \;-\; i\omega\frac{d\tau}{dz} - ik_\beta(m+n)]F_{\omega q}^{(m,n)}(r_x, r_y, \gamma) &= -f_{0\perp}(r)\frac{df_{0\parallel}(\gamma)}{d\gamma} \\ \times \int_0^\infty[\sum_{m',n'}\int_0^\infty\int_0^\infty K_{\omega q}^{(m,n,m',n')}(r_x, r_y|r'_x, r'_y)&F_{\omega q}^{(m',n')}(r'_x, r'_y, \gamma')r'_x dr'_x r'_y dr'_y]d\gamma', \quad (13)\end{aligned}$$

where the symmetric kernel $K_{\omega q}^{(m,n,m',n')}$ is given by

$$\begin{aligned}K_{\omega q}^{(m,n,m',n')} \quad &(r_x, r_y|r'_x, r'_y) = i^{|m|+|n|-(|m'|+|n'|)}(2\pi k_\beta)^2 \\ \times &\int_{-\infty}^{\infty} P_{\omega q}(k_\perp)[\mathrm{J}_{|m|}(k_x r_x)\mathrm{J}_{|n|}(k_y r_y)]\cdot[\mathrm{J}_{|m'|}(k_x r'_x)\mathrm{J}_{|n'|}(k_y r'_y)]d^2\boldsymbol{k}_\perp \quad (14)\end{aligned}$$

and $r = \sqrt{r_x^2 + r_y^2}$ is the amplitude of the electron position in four dimensional transverse phase space.

III. General Solution

By inspecting this equation, it can be seen that the γ dependence of $F^{(m,n)}_{\omega q}$ is such that $F^{(m,n)}_{\omega q} \propto \frac{df_{0\parallel}}{d\gamma}/(q - i\omega\frac{d\tau}{dz} - ik_\beta(m+n))$. It is then useful to define a radial function $R^{(m,n)}_{\omega q}$ as the γ integral of $F^{(m,n)}_{\omega q}$ to eliminate the obvious γ dependence

$$R^{(m,n)}_{\omega q}(r_x, r_y) = \int_0^\infty F^{(m,n)}_{\omega q}(r_x, r_y, \gamma)d\gamma. \tag{15}$$

The integral equation(13) can be solved in a general way as follows[3]. We expand the radial function $R^{(m,n)}_{\omega q}$ using orthogonal functions $f_k^{(|m|,|n|)}(r_x, r_y)$ as

$$R^{(m,n)}_{\omega q}(r_x, r_y) = W_\perp(r)\sum_{k=0}^{\infty} a_k^{(m,n)} f_k^{(|m|,|n|)}(r_x, r_y) r_x^{|m|} r_y^{|n|}. \tag{16}$$

Here, the weight function $W_\perp(r)$ is defined by

$$W_\perp(r) = Cf_{0\perp}(r), \tag{17}$$

where C is a normalization constant to be chosen. The functions $f_k^{(|m|,|n|)}(r_x, r_y)$ are determined so as to satisfy the following orthogonality relationship

$$\int_0^\infty \int_0^\infty W_\perp(r) f_k^{(|m|,|n|)}(r_x, r_y) f_l^{(|m|,|n|)}(r_x, r_y) r_x^{2|m|+1} r_y^{2|n|+1} dr_x dr_y = \delta_{kl}. \tag{18}$$

Using $f_k^{(|m|,|n|)}(r_x, r_y)$, we expand the Bessel functions as

$$\mathrm{J}_{|m|}(k_x r_x)\mathrm{J}_{|n|}(k_y r_y) = \sum_{k=0}^{\infty} C_{|m|,|n|,k}(k_x, k_y) \cdot f_k^{(|m|,|n|)}(r_x, r_y) \cdot r_x^{|m|} r_y^{|n|}, \tag{19}$$

where

$$C_{|m|,|n|,k}(k_x, k_y) = \int_0^\infty \int_0^\infty \mathrm{J}_{|m|}(k_x r_x)\mathrm{J}_{|n|}(k_y r_y) W_\perp(r) f_k^{(|m|,|n|)}(r_x, r_y) r_x^{|m|+1} r_y^{|n|+1} dr_x dr_y. \tag{20}$$

The lowest-order term $C_{|m|,|n|,0}(k_x, k_y)$ has a simpler expression, since the corresponding lowest-order orthogonal function $f_0^{(|m|,|n|)}(r_x, r_y)$ is just a constant. In this case, the integration over the angle $\theta_r = \tan^{-1} r_y/r_x$ can be carried out in Eq.(20), with the result,

$$C_{|m|,|n|,0}(k_x, k_y) = f_0^{(|m|,|n|)} \cos^{|m|}\theta_k \sin^{|n|}\theta_k \int_0^\infty \frac{\mathrm{J}_{|m|+|n|+1}(k_\perp r)}{k_\perp r} W_\perp(r) r^{|m|+|n|+3} dr, \tag{21}$$

where $\theta_k = \tan^{-1} k_y/k_x$.

Inserting Eqs.(16) and (19) into Eq.(13), multiplying by $f_k^{(|m|,|n|)}(r_x, r_y) r_x^{|m|+1} r_y^{|n|+1}$ and integrating over r_x and r_y, we have a matrix equation for the coefficients $a_k^{(m,n)}$:

$$a_k^{(m,n)} + \sum_{m',n',l,j} \beta^{m,n}_{k,l} M^{m,n,l}_{m',n',j} a_j^{(m',n')} = 0, \tag{22}$$

where

$$\beta_{k,l}^{m,n} = \int_0^\infty \int_0^\infty \int_0^\infty \frac{W_\perp(r) f_k^{(|m|,|n|)}(r_x,r_y) f_l^{(|m|,|n|)}(r_x,r_y) r_x^{2|m|+1} r_y^{2|n|+1}}{q - i\omega \frac{d\tau}{dz} - ik_\beta(m+n)} \frac{df_{0\parallel}}{d\gamma} dr_x dr_y d\gamma \quad (23)$$

and the matrix elements are given by

$$M_{m',n',j}^{m,n,l} = i^{|m|+|n|-(|m'|+|n'|)} \frac{(2\pi k_\beta)^2}{C} \int_0^\infty \int_0^\infty P_{\omega q}(k_\perp) C_{|m|,|n|,l}(k_x,k_y) C_{|m'|,|n'|,j}(k_x,k_y) dk_x dk_y. \quad (24)$$

The matrix equation can be symbolically written as

$$(\boldsymbol{I} + \boldsymbol{\beta}\cdot\boldsymbol{M})\boldsymbol{a} = 0, \quad (25)$$

where $\boldsymbol{a}$ is the vector of the coefficient $a_k^{(m,n)}$, $\boldsymbol{I}$ is the unit matrix, and the matrix elements of $\boldsymbol{\beta}$ and $\boldsymbol{M}$ are given by Eqs.(23) and (24), respectively. The nontrivial solution of Eq.(25) requires that

$$det(\boldsymbol{I} + \boldsymbol{\beta}\cdot\boldsymbol{M}) = 0. \quad (26)$$

This dispersion relation gives eigenvalues q as a function of ω or vice versa.

IV. The Lowest-Order Term

It is straightforward to seek zeros of the dispersion relation by computer and the computation requires little cpu time. Numerical studies show a quite rapid convergence of solutions as a function of the matrix size. As a matter of fact, we have found that one can obtain an accurate eigenvalue for the fundamental mode by taking only the lowest-order term $m = n = k = 0$ in both the azimuthal and the radial expansions (see Eqs.(12) and (16)). In this case, an approximate expression for the dispersion relation can be written in a general form as

$$\begin{aligned} 1 &= 4i\frac{k}{k_1}\frac{r_e}{c}\left(\frac{K}{\gamma_r}\right)^2\frac{k_w}{\gamma_r}\int_0^\infty\int_0^\infty \frac{f_{0\parallel}(\gamma)d\gamma}{(q + 2i\frac{k}{k_1}k_w\frac{\gamma-\gamma_r}{\gamma_r} - i\frac{1}{2}kk_\beta^2 r^2)^2} 2\pi^2 k_\beta^2 f_{0\perp}(r) r^3 dr \\ &\times \int_0^{\frac{\pi}{2}} \frac{k^2\theta d\theta}{q + ik_w\frac{k-k_1}{k_1} + i\frac{k\theta^2}{2}} \left(\int_0^\infty 2\pi^2 k_\beta^2 f_{0\perp}(r')\frac{J_1(kr'\theta)}{kr'\theta} r'^3 dr'\right)^2, \end{aligned} \quad (27)$$

where $f_{0\perp}(r)$ is normalized such that $\int_0^\infty 2\pi^2 k_\beta^2 f_{0\perp}(r) r^3 dr = 1$.

Let us investigate the general dispersion relation(27). The integrals over γ and r characterize the Landau damping due to the energy spread and the betatron oscillation via the longitudinal velocity variation. The function in the θ-integral $((\mathrm{J}_1(kr'\theta)/(kr'\theta))^2$ is the well-known diffraction angular distribution of a wave when injected into a screen with a circular hole of radius r'. The integral over θ represents the amount of overlap between the angular distribution of radiation from a single electron and the diffraction angular distribution of the radiation determined by the electron transverse distribution.

In what follows, we write down the above equation in a more specific way for various models of $f_{0\perp}(r)$. Longitudinally, we assume a Gaussian distribution with the rms energy spread, σ_γ.

Hollow Beam : $f_{0\perp}(r) = \frac{1}{(\pi R_0^2 k_\beta)^2}\delta(1-(\frac{r}{R_0})^2)$

$$1 = 2i\frac{k}{k_1}\frac{(2\rho k_w)^3}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-\frac{t^2}{2}}dt}{(q+2i\frac{k}{k_1}k_w\sigma_\gamma t - i\frac{1}{2}kk_\beta^2R_0^2)^2}\int_0^{\frac{\pi}{2}}\frac{(\frac{\mathrm{J}_1(kR_0\theta)}{kR_0\theta})^2(kR_0)^2\theta d\theta}{q+ik_w\frac{k-k_1}{k_1}+i\frac{k\theta^2}{2}} \tag{28}$$

Waterbag Beam : $f_{0\perp}(r) = \frac{2}{(\pi R_0^2 k_\beta)^2}\Theta(1-(\frac{r}{R_0})^2)$

$$1 = 2i\frac{k}{k_1}\frac{(2\rho k_w)^3}{\sqrt{2\pi}}\int_0^1\int_0^\infty \frac{x^3e^{-\frac{t^2}{2}}dxdt}{(q+2i\frac{k}{k_1}k_w\sigma_\gamma t - i\frac{1}{2}kk_\beta^2R_0^2x^2)^2}\int_0^{\frac{\pi}{2}}\frac{2(\frac{4\mathrm{J}_2(kR_0\theta)}{(kR_0\theta)^2})^2(kR_0)^2\theta d\theta}{q+ik_w\frac{k-k_1}{k_1}+i\frac{k\theta^2}{2}} \tag{29}$$

Gaussian Beam : $f_{0\perp}(r) = \frac{1}{(2\pi\sigma_x^2 k_\beta)^2}e^{-\frac{1}{2}(\frac{r}{\sigma_x})^2}$

$$1 = i\frac{k}{2k_1}\frac{(2\rho k_w)^3}{\sqrt{2\pi}}\int_0^\infty\int_0^\infty \frac{e^{-\frac{x^2}{2}}x^3e^{-\frac{t^2}{2}}dxdt}{(q+2i\frac{k}{k_1}k_w\sigma_\gamma t - i\frac{1}{2}kk_\beta^2\sigma_x^2x^2)^2}\int_0^{\frac{\pi}{2}}\frac{e^{-(k\sigma_x\theta)^2}(k\sigma_x)^2\theta d\theta}{q+ik_w\frac{k-k_1}{k_1}+i\frac{k\theta^2}{2}}. \tag{30}$$

In the above equation, ρ is the Pierce parameter[4] defined by $(2\rho k_w)^3 = 2\pi r_e(K/\gamma_r)^2 n_0 k_w/\gamma_r$, n_0 is the charge density, σ_x is the rms transverse beam size, $\Theta(x)$ is the step function $\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ for $x<0$, and we have approximated $[(\mathrm{J}_1(kr_h\theta)/(kr_h\theta))^2 + \mathrm{J}_1'^2(kr_h\theta)]$ by $1/2$, assuming that r_h is much smaller than the beam size.

In the limit of large beam size, $R_0\to\infty$, when $\sigma_\gamma=0$ and $k_\beta=0$, the dispersion relation(28) for the hollow beam will reduce to the well-known cubic equation of the one-dimensional theory. This can be shown briefly as follows. In this limit, Eq.(28) can be approximated by

$$1 = 2i\frac{k}{k_1}\frac{(2\rho k_w)^3}{q^2}\frac{1}{q + ik_w\dfrac{k - k_1}{k_1}}\int_0^\infty (\frac{\mathrm{J}_1(x)}{x})^2 x dx. \tag{31}$$

It follows using $\int_0^\infty (\mathrm{J}_1(x)/x)^2 x dx = 1/2$ that

$$q^2(q + ik_w\frac{k - k_1}{k_1}) - i\frac{k}{k_1}(2\rho k_w)^3 = 0. \tag{32}$$

Introducing $\mu = iq/k_w$, Eq.(32) is the cubic equation of the high gain regime.

In the limit of small beam size, $R_0 \to 0$, when $\sigma_\gamma = 0$ and $k_\beta = 0$, the dispersion relation(28) also gives the correct asymptotic gain derived by Moore[5]. In this limit, the dispersion relation(28) when $k = k_1$ can be approximated by

$$\frac{\hat{g}^2}{2} - \int_0^\infty \frac{x}{x^2 - i\hat{g}\hat{a}^2}(\frac{\mathrm{J}_1(x)}{x})^2 dx = 0, \tag{33}$$

where $\hat{g} = q/((2\rho k_w)^{\frac{3}{2}}(2k_1R_0^2)^{\frac{1}{2}})$ and $\hat{a} = (2\rho k_w)^{\frac{3}{4}}(2k_1R_0^2)^{\frac{3}{4}}$ are Moore's scaled gain and beam size, respectively. By performing the partial integral in Eq.(33) and neglecting the $\log \hat{g}$ term, we obtain Moore's expression

$$\frac{\hat{g}^2}{2} \simeq \frac{1}{4}\log\frac{2}{\hat{a}} \quad \text{or} \quad \hat{g} \simeq \sqrt{\frac{1}{2}\log\frac{2}{\hat{a}}}. \tag{34}$$

As anticipated from the above argument, numerical studies show that the gain $\hat{g}$ obtained by the present dispersion relation(28) is in excellent agreement with that of Moore in the entire range of beam size.

V. Numerical Results

As Krinsky, Yu and Gluckstern[2] have pointed out, the growth rate of the fundamental guided mode can be expressed in a scaled form using only four dimensionless scaling parameters. For the constant beam current case of practical interest, one form of such a scaling relation is

$$\frac{Re(q)}{k_w D} = F(2k_1\varepsilon_x, \frac{\sigma_\gamma}{D}, \frac{k_\beta}{k_w D}, \frac{k - k_1}{k_1 D}), \tag{35}$$

where ε_x is the rms transverse emittance of the electron beam, and D is the scaling parameter defined by

$$D = \sqrt{\frac{2eZ_0}{\pi mc^2}\frac{K^2}{1 + K^2}\frac{I_0}{\gamma_r}}, \tag{36}$$

with I_0 the electron beam current, Z_0 the impedance of free space, e the elementary charge, and m the electron rest mass. Note that D is independent of the model for

$f_{0\perp}(r)$. For the waterbag model, our value of D is smaller than that defined by Krinsky, Yu and Gluckstern[2] by a factor of $\sqrt{2}$.

The solid curves in Fig. 1 show the scaled growth rate $Re(q)/(k_w D)$ as a function of $2k_1\varepsilon_x$ for several values of $k_\beta/(k_w D)$ for the waterbag model. The energy spread σ_γ/D is set to 0, and the detuning is chosen to yield the maximum growth rate. This figure covers most of the practical range of FEL parameters. The dotted curves show the numerical results from Krinsky, Yu and Gluckstern's variational method for the same waterbag model. Good agreement is found. In Fig. 2, we plot $Re(q)/(k_w D)$ against $2k_1\varepsilon_x$ for several values of $k_\beta/(k_w D)$ for the Gaussian model. We choose again $\sigma_\gamma/D = 0$. Comparing Fig. 1 with Fig. 2, we notice that the Gaussian model shows a considerably larger reduction of $Re(q)/(k_w D)$ due to Landau damping for large $k_\beta/(k_w D)$ when $2k_1\varepsilon_x > 1$. This is also the case with the parabolic distribution of $f_{0\perp}(r)$, which gives similar curves to those of the Gaussian model. We also notice that all figures show more or less identical values of $Re(q)/(k_w D)$ for $2k_1\varepsilon_x < 1$. This implies that the emittance or the rms value is a good measure for the beam size when comparing results from different models in the present range of the emittance.

Finally, for a quick estimate of the FEL gain, we present a handy empirical formula of $Re(q)/(k_w D)$ including the energy spread which agrees well with the gains obtained from the dispersion relation(30) for the Gaussian model:

$$\begin{aligned}
\log\frac{Re(q)}{k_w D} &= -\ (\,0.75 + 0.23\Xi + 0.016\Xi^2\,) \\
&\times \Bigg[1 + (2k_1\varepsilon_x\frac{k_\beta}{k_w D})^2/(0.17 + 0.0304\log\frac{k_\beta}{k_w D}) \\
&\quad + (41.34 + 3.69\Xi + 3.62\Xi^2)\cdot((\frac{\sigma_\gamma}{D})^2 + 2.18(\frac{\sigma_\gamma}{D})^4 + 70.9(\frac{\sigma_\gamma}{D})^6)\Bigg]^{\frac{1}{2}}, \qquad (37)
\end{aligned}$$

where

$$\Xi = \log(2k_1\varepsilon_x\frac{k_w D}{k_\beta}). \qquad (38)$$

References

[1] K. J. Kim, Phys. Rev. Lett., 57 (1986) 1871.

[2] S. Krinsky, L. H. Yu, and R. L. Gluckstern, Phys. Rev. Lett., 64 (1990) 3011.

[3] Y. H. Chin, K. Satoh, and K. Yokoya, Part. Accelerators, 13 (1983) 45.

[4] R. Bonifacio, C. Pellegrini, and L. M. Narducci, Opt. Comm., 50 (1984) 373.

[5] G. T. Moore, Opt. Comm., 52 (1984) 46.

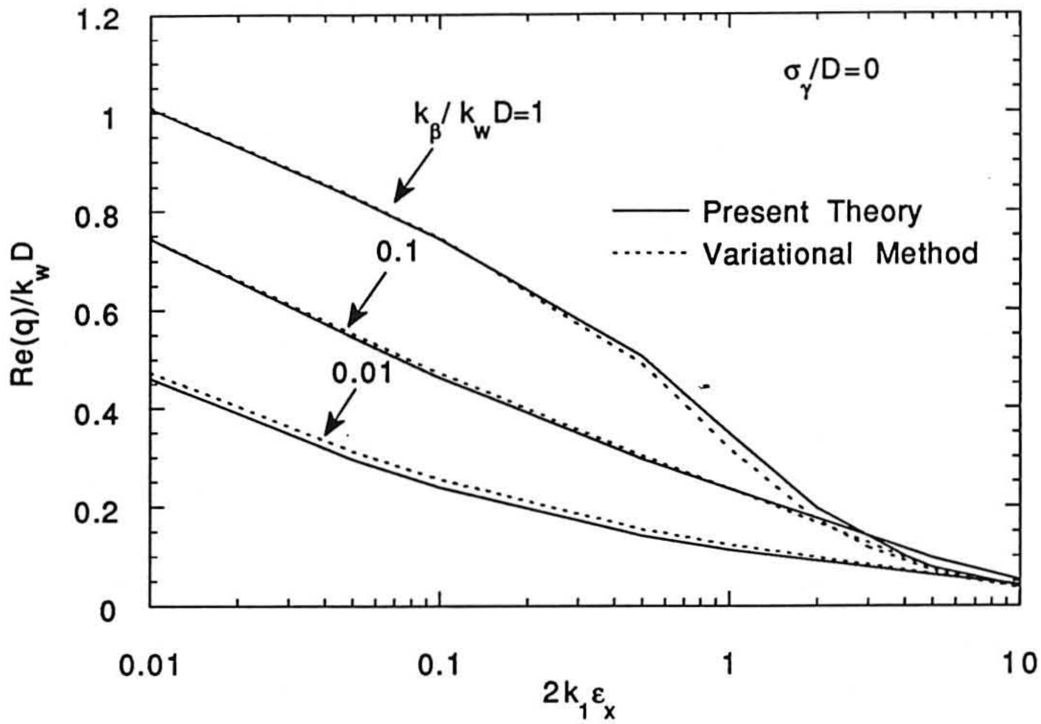


Fig. 1 Scaled growth rate $Re(q)/(k_w D)$ as a function of $2k_1\varepsilon_x$ for several values of $k_\beta/(k_w D)$ for the waterbag model.

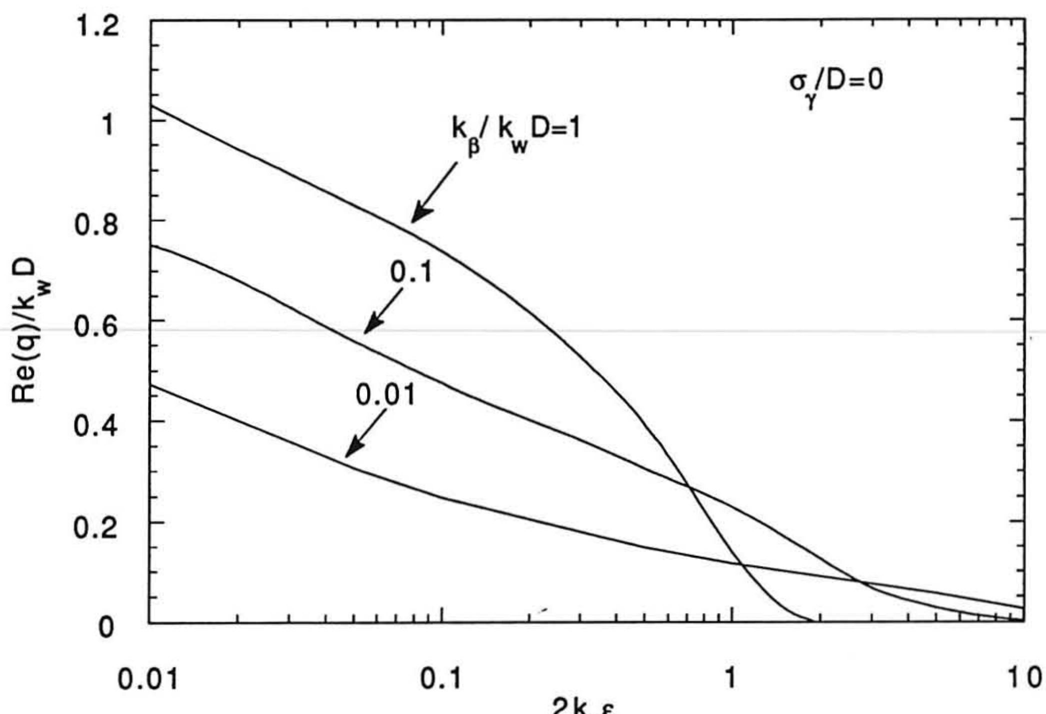


Fig. 2 Scaled growth rate $Re(q)/(k_w D)$ as a function of $2k_1\varepsilon_x$ for several values of $k_\beta/(k_w D)$ for the Gaussian model.